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HOW TO MAKE A TRIANGULATION OF S^3 POLYTOPAL

SIMON A. KING

ABSTRACT. We introduce a numerical isomorphism invariant $p(\mathcal{T})$ for any triangulation $\mathcal{T}$ of S^3. Although its definition is purely topological (inspired by the bridge number of knots), $p(\mathcal{T})$ reflects the geometric properties of $\mathcal{T}$. Specifically, if $\mathcal{T}$ is polytopal or shellable, then $p(\mathcal{T})$ is "small" in the sense that we obtain a linear upper bound for $p(\mathcal{T})$ in the number $n = n(\mathcal{T})$ of tetrahedra of $\mathcal{T}$. Conversely, if $p(\mathcal{T})$ is "small", then $\mathcal{T}$ is "almost" polytopal, since we show how to transform $\mathcal{T}$ into a polytopal triangulation by $O((p(\mathcal{T}))^2)$ local subdivisions. The minimal number of local subdivisions needed to transform $\mathcal{T}$ into a polytopal triangulation is at least $\frac{p(\mathcal{T})}{3n} - n - 2$.

Using our previous results [*The size of triangulations supporting a given link*, Geometry & Topology **5** (2001), 369–398], we obtain a general upper bound for $p(\mathcal{T})$ exponential in n^2. We prove here by explicit constructions that there is no general subexponential upper bound for $p(\mathcal{T})$ in n. Thus, we obtain triangulations that are "very far" from being polytopal.

Our results yield a recognition algorithm for S^3 that is conceptually simpler, although somewhat slower, than the famous Rubinstein–Thompson algorithm.

1. INTRODUCTION

1.1. **Method and results.** A cellular decomposition of the d–dimensional sphere S^d is **polytopal** if it is isomorphic to the boundary complex of a convex $(d+1)$–polytope. The study of polytopal cellular decompositions has a long history and is still an important branch of research. By a theorem of Steinitz [19] of 1922, all triangulations of S^2 are polytopal. However, "most" triangulations of higher-dimensional spheres are not polytopal: Kalai [6] has shown that the number of triangulations of S^d grows faster in the number of vertices than the number of polytopal triangulations of S^d, for $d \geq 4$. Recently, Pfeifle and Ziegler [16] have obtained a similar result for triangulations of S^3. Thus, for a thorough understanding of triangulations of spheres, one has to go beyond the geometric setting of convex polytopes.

This paper is concerned with triangulations $\mathcal{T}$ of the 3–dimensional sphere S^3. We introduce an invariant $d(\mathcal{T})$ of $\mathcal{T}$ that measures to what extent $\mathcal{T}$ fails to be polytopal, and an invariant $p(\mathcal{T})$ that measures "knottedness" of $\mathcal{T}$. We establish a close relationship between $d(\mathcal{T})$ and $p(\mathcal{T})$, thus a relationship between geometric and topological properties of $\mathcal{T}$. The definition of $d(\mathcal{T})$ is based on the following local transformations; compare Figure 1.

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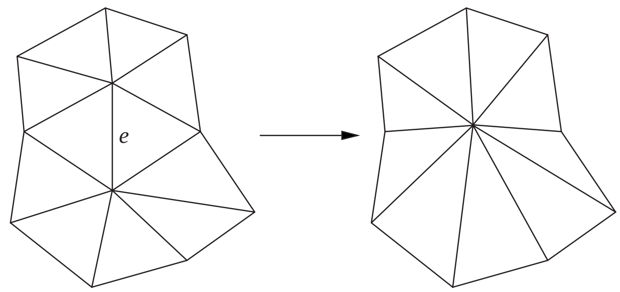

FIGURE 1. Contraction along an edge, in dimension 2

Definition 1.1. Let M be a closed PL-manifold with PL-triangulations $\mathcal{T}_1$ and $\mathcal{T}_2$, and let e be an edge of $\mathcal{T}_1$ with $\partial e = \{a, b\}$. Suppose that $\mathcal{T}_2$ is obtained from $\mathcal{T}_1$ by removing the open star of e and identifying the join $a * \sigma$ with $b * \sigma$ for any simplex σ in the link of e. Then $\mathcal{T}_2$ is obtained from $\mathcal{T}_1$ by the **contraction** along e, and $\mathcal{T}_1$ is obtained from $\mathcal{T}_2$ by an **expansion** along e.

In general, there are edges of $\mathcal{T}_1$ along which the contraction is impossible. This is the case, e.g., if an edge e of $\mathcal{T}_1$ is part of an edge path of length 3 that does not bound a 2–simplex of $\mathcal{T}_1$. Indeed then $\mathcal{T}_2$ has multiple edges and is not a simplicial complex. Obviously any PL–triangulation admits only a finite number of contractions.

Stellar subdivisions of simplices are examples of expansions. Since an expansion increases the number of vertices by one and the number of simplicial complexes with a given number of vertices is finite, it is easy to see that also the number of possible expansions is finite.

By results of Alexander [1] and Moise [15], any two triangulations of a 3–manifold are related by a finite sequence of stellar subdivisions and their inverses. From this, one can conclude that any triangulation $\mathcal{T}$ of S^3 can be transformed into a polytopal triangulation by a finite sequence of expansions. We define $d(\mathcal{T})$ as the length of a shortest sequence of expansions relating $\mathcal{T}$ with a polytopal triangulation. It is a notion of the geometric complexity of $\mathcal{T}$.

Our basic idea is to study $\mathcal{T}$ via its embedded dual graph $\mathcal{C}^1 \subset S^3$, i.e., the 1–skeleton of the dual cellular decomposition $\mathcal{C} = \mathcal{T}^*$ of $\mathcal{T}$. The embedding of $\mathcal{C}^1$ is determined by $\mathcal{T}$ up to ambient isotopy. Thus, any ambient isotopy invariant of $\mathcal{C}^1 \subset S^3$ is an (oriented) isomorphism invariant of $\mathcal{T}$. This allows us to use techniques from knot theory in discrete geometry. Note that if $\mathcal{T}$ is polytopal, then its isomorphism class is already determined by its *abstract* unembedded dual graph; see [21]. However, there are non-polytopal triangulations with isomorphic abstract dual graphs whose embedded dual graphs are not ambiently isotopic to each other.

As the first example of an ambient isotopy invariant of the dual graph of $\mathcal{T}$, we introduce the *polytopality* $p(\mathcal{T})$ of $\mathcal{T}$. We outline its definition. We denote by $\mathcal{C}^i$ the i-skeleton of $\mathcal{C}$, for $i = 0, \ldots, 3$. Let $H : S^2 \times [0,1] \to S^3$ be an embedding in general position to $\mathcal{C}$. A parameter $\xi_0 \in [0,1]$ is a *critical parameter* of H with respect to $\mathcal{C}^i$, if the surface $H(S^2 \times \xi_0)$ is not in general position to $\mathcal{C}^i$. The number of critical parameters of H with respect to $\mathcal{C}^i$ is denoted by $c(H, \mathcal{C}^i)$. The *polytopality* of $\mathcal{T}$ is defined as $p(\mathcal{T}) = \min_H c(H, \mathcal{C}^1)$, where the minimum is taken over all embeddings $H : S^2 \times [0,1] \to S^3$ in general position to $\mathcal{C}$ with $\mathcal{C}^2 \subset H(S^2 \times [0,1])$.

This definition generalizes the bridge number of knots (introduced by Schubert [18] in 1954) to spatial graphs, up to a factor 2. The bridge number of knots formed by edges of $\mathcal{T}$ has been studied earlier, for instance, by Lickorish, Armentrout, Ehrenborg and Hachimori: One obtains linear upper bounds in n for the bridge number of such knots, provided $\mathcal{T}$ is shellable [12], its dual is shellable [2] or $\mathcal{T}$ is vertex decomposable [4]. The following theorem is in the same spirit. For the notions of diagrams and of shellable cell complexes, see [21].

Theorem 1.2. *Let $\mathcal{T}$ be a triangulation of S^3 with n tetrahedra.*

(1) *If $\mathcal{T}$ is polytopal, then $p(\mathcal{T}) = n$.*
(2) *If $\mathcal{T}$ has a diagram, then $p(\mathcal{T}) \leq 3n$.*
(3) *If $\mathcal{T}$ or its dual is shellable, then $p(\mathcal{T}) \leq 7n$.*

It is natural to ask what happens if one drops the geometric assumptions on $\mathcal{T}$. It turns out that the estimates for $p(\mathcal{T})$, and similarly for the bridge number of links formed by edges of $\mathcal{T}$, change dramatically, by the following result.

Theorem 1.3. *For any $m \in \mathbb{N}$ there is a triangulation $\mathcal{T}_m$ of S^3 with at most $856m + 534$ tetrahedra and $p(\mathcal{T}_m) > 2^{m-1}$.*

Hence there is no general subexponential upper bound for the polytopality of a triangulation, in terms of the number of tetrahedra. We obtain the following general bound from our results in [7] that are based on a study of the Rubinstein–Thompson algorithm.

Theorem 1.4. *If $\mathcal{T}$ is a triangulation of S^3 with n tetrahedra, then*

$$n \leq p(\mathcal{T}) < 2^{200n^2}.$$

The definition of $p(\mathcal{T})$ does not rely on geometry. Nevertheless, it turns out that $p(\mathcal{T})$ is in close relationship with geometrical properties of $\mathcal{T}$, as we obtain both lower and upper bounds for $d(\mathcal{T})$ in terms of $p(\mathcal{T})$. We start with a lower bound.

Theorem 1.5. *If $\mathcal{T}$ is a triangulation of S^3 with n tetrahedra, then*

$$d(\mathcal{T}) > \frac{p(\mathcal{T})}{2n+1} - n - \frac{5}{3}.$$

This together with Theorem 1.3 implies that there are triangulations of S^3 that are "very far" from being polytopal. So in a certain sense it reflects Pfeifle's and Ziegler's result [16] that "most" triangulations of S^3 are not polytopal.

To obtain upper bounds for $d(\mathcal{T})$ in $p(\mathcal{T})$, we consider the class of **edge contractible** triangulations. A triangulation of S^d is edge contractible if one can transform it into the boundary complex of a $(d+1)$–simplex by successive contractions along edges. It is well known and easy to show that any edge contractible triangulation is polytopal. All triangulations of S^2 are edge contractible by a theorem of Wagner [20]. In [11], Section 6.3, one finds an example of a polytopal triangulation of S^3 that is not edge contractible.

Theorem 1.6. *From any triangulation $\mathcal{T}$ of S^3, one can obtain an edge contractible triangulation of S^3 by a sequence of at most $512(p(\mathcal{T}))^2 + 869p(\mathcal{T}) + 376$ successive expansions. In particular, if n is the number of tetrahedra,*

$$d(\mathcal{T}) \leq 512(p(\mathcal{T}))^2 + 869p(\mathcal{T}) + 376 < 2^{401n^2}.$$

The bounds in Theorem 1.6 are certainly not sharp. Theorem 1.2 implies that under additional geometric or combinatorial assumptions on $\mathcal{T}$ one can replace the estimate of Theorem 1.6 by a quadratic bound in n. However, a general upper bound for $d(\mathcal{T})$ must be at least exponential in n, by Theorems 1.3 and 1.5. Recently we applied Theorem 1.6 to prove an upper bound for the *crossing number* of links embedded in the 1–skeleton of $\mathcal{T}$; see [9].

According to Theorems 1.2 and 1.5, if $\mathcal{T}$ is geometrically "simple", then $p(\mathcal{T})$ is small. According to Theorem 1.6, if $p(\mathcal{T})$ is small, then $\mathcal{T}$ is not far from being polytopal. Thus $p(\mathcal{T})$ turns out to be a measure for the geometric complexity of $\mathcal{T}$.

As an intermediate step in the proof of Theorem 1.6, we establish the following linear upper bound for the "distance" of two triangulations of S^3.

Theorem 1.7. *Any two triangulations $\mathcal{T}, \widetilde{\mathcal{T}}$ of S^3 are related by a sequence of at most $325\left(p(\mathcal{T}) + p(\widetilde{\mathcal{T}})\right) + 508$ expansions and contractions.*

This together with Theorem 1.4 implies that any two triangulations of S^3 with $\leq n$ tetrahedra can be related by a sequence of less than 2^{201n^2} contractions and expansions. Mijatović [14] obtained a similar estimate concerning bistellar moves rather than contractions and expansions, using a modified form of our results in [7]. Under additional geometric or combinatorial assumptions on $\mathcal{T}$, Theorem 1.7 together with Theorem 1.2 yields a linear bound in n. Recently we have extended our method to the projective space $P^3_{\mathbb{R}}$, proving that any two triangulations of $P^3_{\mathbb{R}}$ with at most n tetrahedra can be related by a sequence of less than $2^{27000\,n^2}$ edge contractions and expansions [10].

Theorem 1.7 yields the following simple try-and-check algorithm for the recognition of S^3. Let $\mathcal{T}$ be a triangulation of a closed 3–manifold N with n tetrahedra. Try all sequences of $< 2^{201n^2}$ contractions and expansions starting from $\mathcal{T}$, which are finite in number. If one of them transforms $\mathcal{T}$ into the boundary complex of a 4–simplex, then N is homeomorphic to S^3. Otherwise, N is not homeomorphic to S^3 by Theorem 1.7. This recognition algorithm is certainly slower, but simpler, than the Rubinstein–Thompson recognition algorithm.

1.2. **Outline and organization of the proofs.** We outline the proof of Theorem 1.2. Let $\mathcal{T}$ be a triangulation of S^3, and let $\mathcal{C}$ be its dual cellular decomposition. If $\mathcal{T}$ is polytopal (resp. $\mathcal{T}$ has a diagram), then $\mathcal{C}$ (resp. its barycentric subdivision $\mathcal{C}'$) has a diagram. A sweep-out of $\mathbb{R}^3$ by Euclidean planes in general position to the diagram of $\mathcal{C}$ (resp. $\mathcal{C}'$) has only critical points in the vertices of the diagram. Hence $p(\mathcal{T})$ is bounded by the number of vertices (resp. the number of vertices plus the number of edges) of $\mathcal{C}$. If $\mathcal{T}$ or $\mathcal{C}$ is shellable, then $\mathcal{C}'$ is shellable. Using a shelling order of the tetrahedra of $\mathcal{C}'$, we construct an embedding $H\colon\ S^2 \times I \to S^3$ that has at most one critical point in each open simplex of $\mathcal{C}'$. This yields a bound for $p(\mathcal{T})$.

Theorem 1.3 is based on a result of Hass, Snoeyink and Thurston [5]. We construct for any $m \in \mathbb{N}$ a simple cellular decomposition $\mathcal{Z}_m$ of the solid torus with a linear bound in m for the number of vertices, so that any meridional disc for the torus intersects $\mathcal{Z}^1_m$ in $\geq 2^{m-1}$ points. We glue two copies of $\mathcal{Z}_m$ together and obtain a simple cellular decomposition $\mathcal{C}_m$ of S^3 that is dual to a triangulation $\mathcal{T}_m$

of S^3. It follows $p(\mathcal{T}_m) > 2^{m-1}$, since for any sweep-out there is a level sphere that contains a meridional disc in one of the solid tori.

We outline the proof of Theorem 1.5. Let $\mathcal{T}$ be a triangulation of S^3 with n tetrahedra. Let $\tilde{\mathcal{T}}$ be a polytopal triangulation that is obtained from $\mathcal{T}$ by a sequence of $d(\mathcal{T})$ expansions. It has at most $n+d(\mathcal{T})$ vertices. Since $\tilde{\mathcal{T}}$ is polytopal, there is an embedding $H\colon S^2 \times I \to S^3$ with $\tilde{\mathcal{T}}^2 \subset H(S^2 \times I)$ that has only critical points in the vertices of $\tilde{\mathcal{T}}$, i.e., $c(H, \tilde{\mathcal{T}}^1) \le n+d(\mathcal{T})$. We have $c(H, \mathcal{T}^1) \le c(H, \tilde{\mathcal{T}}^1)$, since $\mathcal{T}^1 \subset \tilde{\mathcal{T}}^1$. We finish the proof of Theorem 1.5 by giving a lower bound for $c(H, \mathcal{T}^1)$ in terms of $p(\mathcal{T})$ and n. This lower bound also yields Theorem 1.4 from our results in [7].

The proofs of Theorems 1.6 and 1.7 are based on the interplay of cellular structures on S^3 and isotopies of surfaces. Let $\mathcal{T}$ be a triangulation of S^3, and let $\mathcal{C}$ be its dual cellular decomposition. Let $H\colon S^2 \times I \to S^3$ be an embedding in general position with respect to $\mathcal{C}$ such that $\mathcal{C}^2 \subset H(S^2 \times I)$ and $c(H, \mathcal{C}^1) = p(\mathcal{T})$. In the first step of the proof of Theorem 1.7, we change H by canceling pairs of critical parameters so that $c(H, \mathcal{C}^2)$ is bounded in terms of $p(\mathcal{T})$. We associate to any non-critical parameter $\xi \in I$ a 2–dimensional polyhedron $P_\xi \subset S^3$ that consists of $H(S^2 \times \xi)$ and parts of $\mathcal{C}^2$. It changes by insertions and deletions of 2–strata, when ξ passes a critical parameter of H with respect to $\mathcal{C}^2$. We can bound the number of vertices of the inserted or deleted 2–strata.

The polyhedron P_ξ is the 2–skeleton of a cellular decomposition of S^3, whose barycentric subdivision is a triangulation $\mathcal{T}_\xi$. The deletion of a 2–stratum c in P_ξ (resp. its insertion into P_ξ) gives rise to a sequence of contractions (resp. expansions) starting at $\mathcal{T}_\xi$. The number of contractions (resp. expansions) is determined by the number of vertices in ∂c. We obtain a sequence of expansions and contractions that relates $\mathcal{T}$ with the boundary complex of a 4–simplex, whose length is bounded in terms of $p(\mathcal{T})$. This yields Theorem 1.7.

The proof idea for Theorem 1.6 is to attach 2–strata to P_ξ as in the proof of Theorem 1.7, when ξ passes a critical parameter of H, and to postpone the deletions of 2–strata until all insertions are done. This gives rise to a sequence of expansions followed by a sequence of contractions that relate $\mathcal{T}$ with the boundary complex of a 4–simplex. Here the boundary of the inserted 2–strata may contain additional vertices, at the points of intersection with the boundary of other inserted (and not yet deleted) 2–strata. This yields the quadratic bound in Theorem 1.6. The bound in terms of n is a consequence of Theorem 1.4.

The paper is organized as follows. In Subsection 2.1, we define the polytopality and prove Theorems 1.2, 1.4 and 1.5. Subsection 2.2 is devoted to the proof of Theorem 1.3. Theorems 1.6 and 1.7 are proven in Section 3. In Subsection 3.1, we change any embedding $H\colon S^2 \times I \to S^3$ into an embedding $\tilde{H}\colon S^2 \times I \to S^3$ with an upper bound for $c(\tilde{H}, \mathcal{C}^2)$ in terms of $c(H, \mathcal{C}^1) = c(\tilde{H}, \mathcal{C}^1)$. The proofs of Theorems 1.7 and 1.6 are finished in Subsections 3.2 and 3.3, respectively.

2. Polytopality

2.1. **Definition and upper bounds.** After exposing some technical notions, we introduce in this subsection a numerical invariant for a triangulation $\mathcal{T}$ of S^3, called *polytopality*. It is an ambient isotopy invariant of the embedded dual graph of $\mathcal{T}$ and is inspired by the bridge number of links due to Schubert [18]. We prove a general upper bound for $p(\mathcal{T})$ in terms of the number of tetrahedra, based on our results in [7]. Under different hypotheses on $\mathcal{T}$, we obtain stronger bounds. We prove a lower bound for $d(\mathcal{T})$ in terms of $p(\mathcal{T})$.

Let M be a closed 3–manifold. For a cellular decomposition $\mathcal{Z}$ of M and for $i = 0, \dots, 3$, let $\mathcal{Z}^i$ denote the i–skeleton of $\mathcal{Z}$. In this paper we only consider cellular decompositions such that if $\varphi\colon B^k \to M$ is the attaching map of a k–cell, then for any open cell c the restriction of φ to a connected component of $\varphi^{-1}(c)$ is a homeomorphism onto c. We do not assume that the closure of an open k–cell is a compact k–ball. Recall that a **triangulation** of M is a cellular decomposition of M that forms a simplicial complex. We denote by $\#(X)$ the number of connected components of a topological space X. The notation $X \subset M$ stands for a tame embedding of X into M, and $U(X)$ denotes an open regular neighborhood of X in M. We denote the unit interval by $I = [0, 1]$.

Let $\mathcal{Z} \subset M$ be a 2–dimensional cell complex (later, it will be the 1– or 2–skeleton of a triangulation of M or of its dual). An **isotopy mod** $\mathcal{Z}$ is an ambient isotopy that preserves each open cell of $\mathcal{Z}$ as a set. Let S be a closed surface. Let $H\colon S \times I \to M$ be an embedding. For $\xi \in I$, set $H_\xi = H(S \times \xi)$. A number $\xi \in I$ is a **critical parameter** of H with respect to $\mathcal{Z}$, and a point $p \in H_\xi$ is a **critical point** of H with respect to $\mathcal{Z}$, if p is a vertex of $\mathcal{Z}$, a point of tangency of H_ξ with $\mathcal{Z}^1$, or a point of tangency of H_ξ with $\mathcal{Z}^2$.

An embedding $H\colon S \times I \to M$ is $\mathcal{Z}^1$–**Morse** if it has finitely many critical parameters with respect to $\mathcal{Z}$, to each critical parameter of H with respect to $\mathcal{Z}$ belongs exactly one critical point, and one connected component of $U(p_0) \setminus H_{\xi_0}$ is disjoint from $\mathcal{Z}^1$, for each critical point $p_0 \in \mathcal{Z}^1 \setminus \mathcal{Z}^0$. If H is a $\mathcal{Z}^1$–Morse embedding, then $c(H, \mathcal{Z}^i)$ denotes the number of critical points of H in $\mathcal{Z}^i$, for $i = 1, 2$.

Definition 2.1. Let $\mathcal{T}$ be a triangulation of S^3, and let $\mathcal{C}$ be its dual cellular decomposition. The **polytopality** of $\mathcal{T}$ is the number

$$p(\mathcal{T}) = \min_H c(H, \mathcal{C}^1),$$

where the minimum is taken over all $\mathcal{C}^1$–Morse embeddings $H\colon S^2 \times I \to S^3$ with $\mathcal{C}^2 \subset H(S^2 \times I)$.

The following lemma allows us to reduce Theorem 1.4 to our results in [7], and is also useful in proving Theorem 1.5.

Lemma 2.2. *Let $\mathcal{T}$ be a triangulation of S^3 with n tetrahedra, and let $H\colon S^2 \times I \to S^3$ be a $\mathcal{T}^1$–Morse embedding with $\mathcal{T}^2 \subset H(S^2 \times I)$. Then*

$$p(\mathcal{T}) < (2n + 1) \cdot c(H, \mathcal{T}^1) + 5n.$$

Proof. Let $\mathcal{T}'$ be the barycentric subdivision of $\mathcal{T}$. All regular neighborhoods occuring in this proof are to be understood with respect to $\mathcal{T}$. For any simplex σ of $\mathcal{T}$, choose a vertex v_σ of σ. Let $p_\sigma \in (\mathcal{T}')^0$ be the barycenter of σ. By ambient isotopy of $(\mathcal{T}')^1$ with support in $U(\sigma)$, we can assume that $p_\sigma \in U(v_\sigma)$.

Let τ be a boundary simplex of σ, and let e be the edge of $\mathcal{T}'$ with endpoints p_σ, p_τ. If $v_\sigma = v_\tau$, then we can assume by ambient isotopy of $(\mathcal{T}')^1$ with support in $U(\sigma)$ that $e \subset U(v_\sigma)$ and H has no critical points in the interior of e. If $v_\sigma \neq v_\tau$, then let f be the edge of $\mathcal{T}$ with endpoints $v_\sigma, v_\tau \in \partial\sigma$. We can assume by ambient isotopy of $(\mathcal{T}')^1$ with support in $U(\sigma)$ that $e \subset U(f)$ and that the critical points of H in the interior of e are in bijective correspondence to those in the interior of

f. The latter case occurs at most $2n+1$ times for any edge of $\mathcal{T}$, since the star of e in $\mathcal{T}$ contains $\leq 2n+1$ simplices. Thus H has $\leq (2n+1)\cdot(c(H,\mathcal{T}^1)-\#(\mathcal{T}^0))$ critical points in $(\mathcal{T}')^1 \setminus (\mathcal{T}')^0$. The 1–skeleton of the dual cellular decomposition of $\mathcal{T}$ is contained in $(\mathcal{T}')^1$. Thus

$$p(\mathcal{T}) \leq c(H,(\mathcal{T}')^1) \leq (2n+1)\cdot\left(c(H,\mathcal{T}^1)-\#(\mathcal{T}^0)\right) + \#\left((\mathcal{T}')^0 \setminus \mathcal{T}^0\right).$$

Since $\mathcal{T}$ has n tetrahedra, $2n$ 2–simplices and at most $2n$ edges, $(\mathcal{T}')^0 \setminus \mathcal{T}^0$ comprises $\leq 5n$ vertices. □

Proof of Theorem 1.4. Let $\mathcal{T}$ be a triangulation of S^3 with n tetrahedra and let $\mathcal{C}$ be its dual cellular decomposition. Since the vertices of $\mathcal{C}$ are critical points, it follows that $p(\mathcal{T}) \geq n$. By our results in [7], that we have slightly improved in Chapters 3 and 4 of [8], there is a $\mathcal{T}^1$–Morse embedding $H\colon S^2\times I \to S^3$ with $\mathcal{C}^2 \subset H(S^2\times I)$ and $c(H,\mathcal{T}^1) < 2^{190n^2}$. By Lemma 2.2, we have

$$p(\mathcal{T}) < (2n+1)\cdot c(H,\mathcal{T}^1) + 5n < 2^{200n^2}. \qquad \square$$

Proof of Theorem 1.5. Let $\mathcal{T}$ be a triangulation of S^3 with n tetrahedra, and let $\tilde{\mathcal{T}}$ be a polytopal triangulation with $\tilde{v}$ vertices that is obtained from $\mathcal{T}$ by $d(\mathcal{T})$ expansions. Since $\tilde{\mathcal{T}}$ has a diagram, there is a $\tilde{\mathcal{T}}^1$–Morse embedding $H\colon S^2\times I \to S^3$ such that $\tilde{\mathcal{T}}^1 \subset H(S^2\times I)$ and $c(H,\tilde{\mathcal{T}}^1)$ equals the number of vertices of $\tilde{\mathcal{T}}$. Since $\mathcal{T}^1 \subset \tilde{\mathcal{T}}^1$ and by Lemma 2.2, we have

$$\tilde{v} = c(H,\tilde{\mathcal{T}}^1) \geq c(H,\mathcal{T}^1) > \frac{p(\mathcal{T})-5n}{2n+1}.$$

The number v of vertices of $\mathcal{T}$ is bounded from above by n. Since $d(\mathcal{T}) = \tilde{v} - v$, we obtain

$$d(\mathcal{T}) > \frac{p(\mathcal{T})}{2n+1} - n - \frac{5}{3}. \qquad \square$$

The rest of this section is devoted to the proof of Theorem 1.2. The three separate claims are proved in the following three lemmas. Recall that a d–**diagram** is a decomposition of a convex d–polytope into convex polytopes; see [21] for details. A cellular decomposition of S^{d+1} has a diagram, if by removing one of its top-dimensional cells it becomes isomorphic to a d–diagram. It is well known that any polytopal cellular decomposition of S^d has a so-called *Schlegel diagram* (named after Schlegel [17]).

Lemma 2.3. *Let $\mathcal{T}$ be a triangulation $\mathcal{T}$ of S^3 with n tetrahedra. If $\mathcal{T}$ is polytopal, then $p(\mathcal{T}) = n$.*

Proof. Let $\mathcal{C}$ be the dual cellular decomposition of $\mathcal{T}$. Since $\mathcal{T}$ is polytopal, $\mathcal{C}$ is polytopal as well. Thus $\mathcal{C}$ has a Schlegel diagram $\mathcal{D} \subset \mathbb{R}^3 = S^3 \setminus \{\infty\}$. We choose coordinates (x,y,z) for $\mathbb{R}^3$ so that no edge of $\mathcal{D}$ is parallel to the xy–plane. Then a sweep-out of $\mathbb{R}^3$ by planes parallel to the xy–plane gives rise to a $\mathcal{C}^1$–Morse embedding having only critical points in the n vertices of $\mathcal{C}$, i.e., $p(\mathcal{T}) = n$. □

Lemma 2.4. *Let $\mathcal{T}$ be a triangulation $\mathcal{T}$ of S^3 with n tetrahedra. If $\mathcal{T}$ has a diagram, then $p(\mathcal{T}) \leq 3n$.*

Proof. Let $\mathcal{T}'$ be the barycentric subdivision of $\mathcal{T}$. Let $\Gamma \subset (\mathcal{T}')^1$ be the 1–skeleton of the dual cellular decomposition of $\mathcal{T}$. Since $\mathcal{T}$ has a diagram, $\mathcal{T}'$ also has a diagram. A sweep-out by Euclidean planes yields a $(\mathcal{T}')^1$–Morse embedding

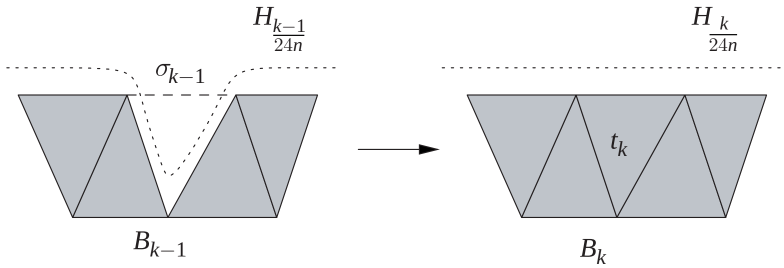

FIGURE 2. Sweep-out along a shelling

$H\colon S^2 \times I \to S^3$ with $(\mathcal{T}')^0$ as set of critical points. The critical points of H in Γ are thus the n vertices of Γ and at most one point in each of the $2n$ open edges of Γ. This yields the lemma. □

Lemma 2.5. *Let $\mathcal{T}$ be a triangulation of S^3 with n tetrahedra. If $\mathcal{T}$ or its dual is shellable, then $p(\mathcal{T}) \leq 7n$.*

Proof. Let $\mathcal{T}'$ be the barycentric subdivision of $\mathcal{T}$. Since $\mathcal{T}$ or its dual is shellable, $\mathcal{T}'$ is also shellable. Thus there is a shelling order $t_1, \ldots, t_{24n}$ on the open tetrahedra of $\mathcal{T}'$ such that $B_k = \bigcup_{i=1}^k \overline{t_i}$ is a compact 3–ball for all $k = 1, \ldots, 24n-1$.

Let $k = 1, \ldots, 24n-2$. If $\partial t_{k+1} \cap B_k$ contains exactly j open 2–simplices of $\mathcal{T}'$, then there is a unique open $(3-j)$-simplex $\sigma_k \subset \partial t_{k+1} \setminus B_k$ of $\mathcal{T}'$. There is a $(\mathcal{T}')^1$–Morse embedding $H\colon S^2 \times I \to S^3$ with the following properties:

(1) $H_0 \subset t_1$,
(2) $H_{\frac{1}{24n}} = \partial U(B_1)$ and H has exactly four critical parameters in $[0, \frac{1}{24n}]$ with respect to $(\mathcal{T}')^1$, with critical points in the vertices of t_1, and
(3) $H_{\frac{k}{24n}} = \partial U(B_k)$ for any $k = 2, \ldots 24n-1$, and H has exactly one critical parameter in $[\frac{k-1}{24n}, \frac{k}{24n}]$ with respect to $(\mathcal{T}')^2$, namely with critical point in the barycenter of σ_{k-1}. Compare the 2–dimensional sketch in Figure 2.

By construction, H has at most one critical point in each open simplex of $\mathcal{T}'$, namely in its barycenter. Thus the critical points of H with respect to the 1–skeleton of the dual cellular decomposition of $\mathcal{T}$ are its n vertices and at most three critical points in each of its $2n$ open edges. This yields the lemma. □

2.2. **The polytopality grows exponentially.** This section is devoted to the proof of Theorem 1.3. We outline the construction after recalling a couple of notions. A 2–polyhedron Q is **simple** or a **fake surface** if the link of any point in Q is homeomorphic to (i) a circle or (ii) a circle with diameter or (iii) a complete graph with four vertices. A 2–**stratum** of Q is a connected component of the union of points of type (i). The points of type (iii) are the **intrinsic vertices** of Q. A cellular decomposition $\mathcal{C}$ of a compact 3–manifold is simple if $|\mathcal{C}^2|$ is simple and $\mathcal{C}^0$ is a union of intrinsic vertices of $|\mathcal{C}^2|$. This corresponds to the classical notion of simple convex polytopes [21].

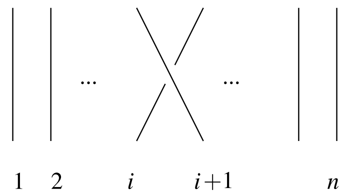

FIGURE 3. The braid generator σ_i

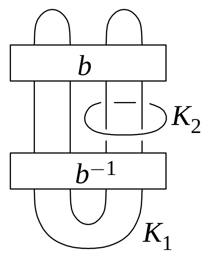

FIGURE 4. The link $K_1 \cup K_2$

For $n \in \mathbb{N}$, let $\mathcal{B}_n$ denote the group of braids with n strands; see [3], for instance. It is generated by $\sigma_1, \ldots, \sigma_{n-1}$, where σ_i corresponds to a crossing of the i–th over[1] the $(i+1)$–th strand of the braid; see Figure 3.

Let $b = \sigma_{i_k}^{\epsilon_k} \cdot \sigma_{i_{k-1}}^{\epsilon_{k-1}} \cdots \sigma_{i_1}^{\epsilon_1} \in \mathcal{B}_4$ be a braid with k crossings, where $\epsilon_j \in \{+1, -1\}$ for $j = 1, \ldots, k$. Let $K_1 \cup K_2 \subset S^3$ be the link that is defined in Figure 4. Both K_1 and K_2 are unknots. We will use a result of Hass, Snoeyink and Thurston [5], providing a sequence of examples of a braid b such that any spanning disc for K_1 intersects K_2 in at least an exponential number of points, in terms of k. Let $V = S^3 \setminus U(K_1)$, which is a solid torus containing K_2 as a not necessarily trivial knot.

We explain the proof idea for Theorem 1.3. We start by constructing a cellular decomposition $\mathcal{Z}_b$ of V with $K_2 \subset \mathcal{Z}_b^1$. For the construction of $\mathcal{Z}_b$, we put together the bricks shown in Figures 5, 7, 9 and 11 (see Construction 2.6), and then drill out a regular neighborhood of K_1 (see Lemma 2.7). Next we glue two modified copies of $\mathcal{Z}_b$ together (see Construction 2.9) in order to obtain a simple cellular decomposition of S^3 that is dual to a triangulation (see Lemma 2.10). If one chooses b according to [5], then the polytopality of the triangulation is "very big", yielding Theorem 1.3.

Construction 2.6. Let (x, y, z) be coordinates for $\mathbb{R}^3 = S^3 \setminus \{\infty\}$. We define $W = \{0 \le x \le 5, -k-2 \le y \le k+2, -1 \le z \le 1\}$ and

$$\begin{aligned} X &= \{z = 0\} \cup \{x \in \mathbb{Z}, z \ge 0\} \cup \{y = -\frac{1}{2}, z \le 0\} \\ &\quad \cup \{2 \le x \le 3, y = 0, z \ge 0\} \cup \{2 \le x \le 3, y = -k - \frac{3}{2}, z \ge 0\}. \end{aligned}$$

[1] Note that in the literature appear different conventions on whether in σ_i the i–th strand crosses *over* or *under* the $(i+1)$–th strand.

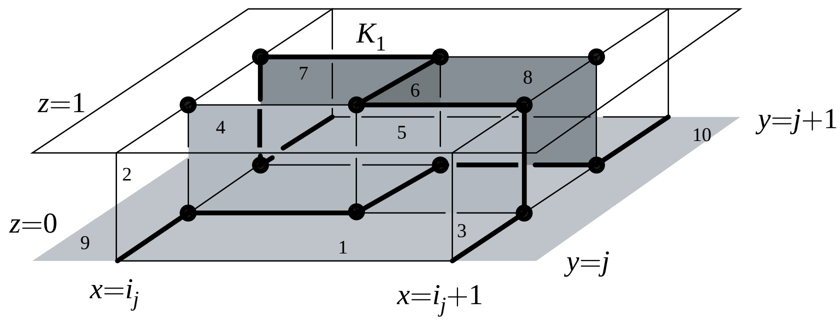

FIGURE 5. Realization of the crossing σ_{i_j} of b

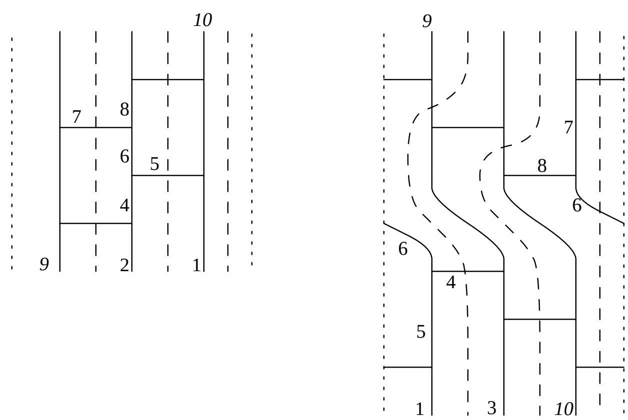

FIGURE 6. ∂V near a crossing of b

The simple 2–polyhedron $P = \partial W \cup (X \cap W) \cup (\{y = \frac{1}{2}\} \setminus W)$ has 28 intrinsic vertices. The unbounded 2–stratum $\{y = \frac{1}{2}\} \setminus W$ is considered as a disc in S^3. We define

$$\begin{aligned} P_b = \quad P \quad & \cup \bigcup_{j=1}^{k} \Big(\{i_j \le x \le i_j + 1, \pm y = j + \frac{1}{3}, 0 \le z \le 1\} \\ & \cup \{i_j \le x \le i_j + 1, \pm y = j + \frac{2}{3}, 0 \le z \le 1\} \\ & \cup \{x = i_j + \frac{1}{2}, j + \frac{1}{3} \le \pm y \le j + \frac{2}{3}, 0 \le z \le 1\} \Big). \end{aligned}$$

One observes that P_b is the 2–skeleton of a simple cellular decomposition of S^3 with $24k + 28$ vertices that is dual to a triangulation. Any crossing of the braid bb^{-1} is realized in P_b^1; see Figure 5. The figure shows the crossing $\sigma_{i_j}^{+1}$ of b, where b is bold. Here and in all subsequent figures, thick dots indicate intrinsic vertices of simple 2–polyhedra. Also the "caps" and "cups" that yield K_1 from bb^{-1} are realized in P_b^1; see Figures 7 and 9. Thus we can assume $K_1 \subset P_b^1$. □

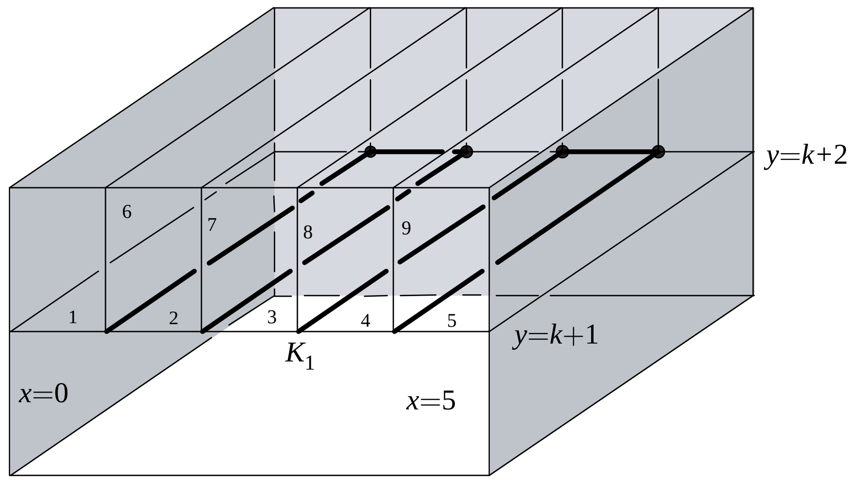

FIGURE 7. Realization of the caps of b

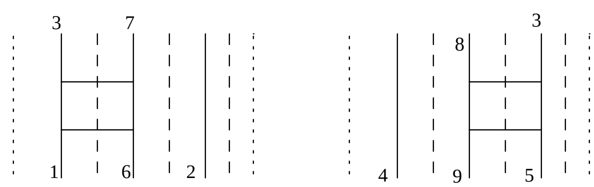

FIGURE 8. ∂V in the caps of b

Lemma 2.7. *The* 2*–polyhedron* $Q_b = (P_b \setminus U(K_1)) \cup \partial U(K_1) \subset V$ *is the* 2*–skeleton of a simple cellular decomposition* $\mathcal{Z}_b$ *of* V *with* $< 48k + 56$ *vertices that is dual to a triangulation of* V*. A representative of* $K_2 \subset V$ *is formed by* 11 *edges of* $\mathcal{Z}_b$*.*

Proof. Since P_b is the 2–skeleton of a cellular decomposition S^3 dual to a triangulation, any connected component of $V \setminus Q_b$ is a ball, the intersection of any two of these compact balls is connected, and the 2–strata of Q_b in the interior of V are discs.

We show that the closure of any 2–stratum of Q_b in ∂V is a disc, and the intersection of any two of these discs is connected. A 2–stratum of Q_b in ∂V is a connected component of $\partial U(K_1) \setminus P_b$. Since K_1 is not contained in the boundary of a single connected component of $S^3 \setminus P_b$, the closure of any component of $\partial U(K_1) \setminus P_b$ is a disc. Since K_1 is not a union of two arcs that are each contained in the boundary of a connected component of $S^3 \setminus P_b$, the intersection of the closures of two connected components of $\partial U(K_1) \setminus P_b$ is connected.

Therefore Q_b is the 2–skeleton (including ∂V) of a simple cellular decomposition $\mathcal{Z}_b$ of V, and the dual of $\mathcal{Z}_b$ has no loops or multiple edges, thus, is a triangulation. Any vertex of P_b in K_1 gives rise to two vertices of $\mathcal{Z}_b$. Thus $\mathcal{Z}_b$ has $< 48k + 56$ vertices. The 1–skeleton of $\mathcal{Z}_b$ contains K_2 as a path of 11 edges, namely seven

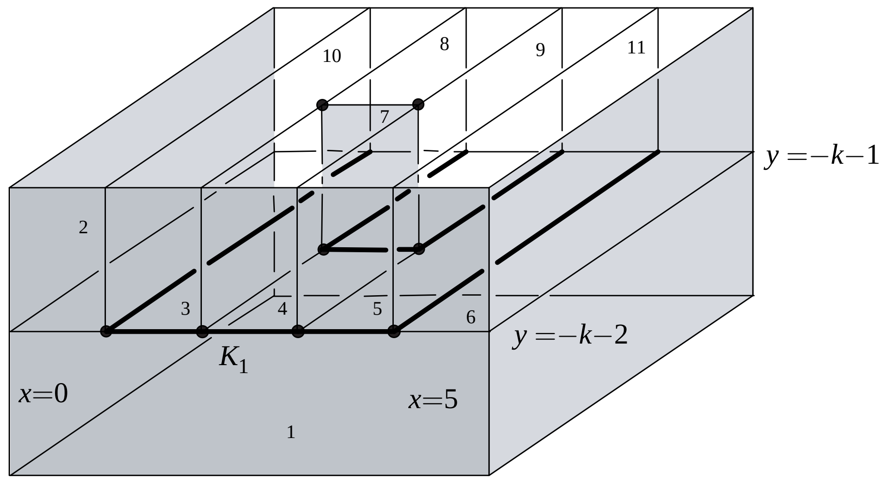

FIGURE 9. Realization of the cups of b

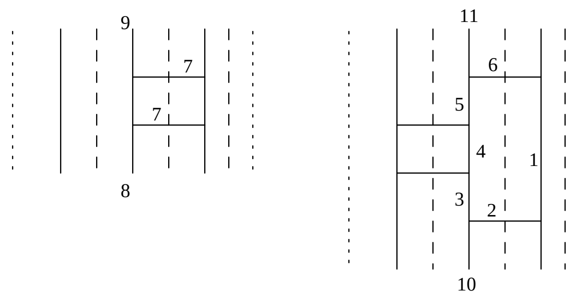

FIGURE 10. ∂V in the cups of b

edges corresponding to edges of P_b (the thick dotted lines in Figure 11) and four edges in $P_b \cap \partial U(K_1) \subset \mathcal{Z}_b^1$ (the thin dotted lines in the figure). □

Our plan is to glue two copies of $\mathcal{Z}_b$ together, to obtain a simple cellular decomposition of S^3. In order to keep a bound for the number of vertices that is linear in k, we need an "adaptor" between the two copies of $\mathcal{Z}_b$. The construction of the adaptor is based on the following lemma.

Lemma 2.8. *There is a disjoint union $\Lambda \subset \partial V$ of three meridians of V such that $\#(\Lambda \cap \mathcal{Z}_b^1) = 22k + 16$ and the intersection of any component of $\partial V \setminus \Lambda$ with any 2–cell of $\mathcal{Z}_b$ in ∂V is connected.*

Proof. We construct $\Lambda \subset \partial V$ according to Figures 6, 8, 10 and 12. Figure 6 shows the two annuli contained in ∂V that correspond to the two sub-arcs of K_1 shown in Figure 5; the annuli are cut along the dotted lines (left and right side of the rectangles in the figure). The figure shows the pattern of $\partial V \cap \mathcal{Z}_b^1$, where the numbers in Figure 6 at the edges of $\mathcal{Z}_b$ in ∂V correspond to the numbers in Figure 5 at the 2–strata of P_b. The broken lines indicate Λ. One sees 11 points of $\Lambda \cap \mathcal{Z}_b^1$. Thus the $2k$ crossings of bb^{-1} contribute to $22k$ points in $\Lambda \cap \mathcal{Z}_b^1$.

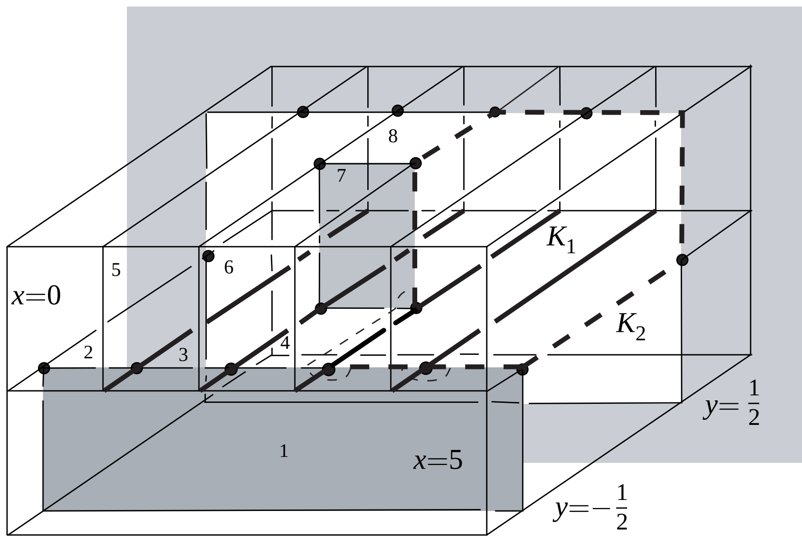

FIGURE 11. Realization of K_2

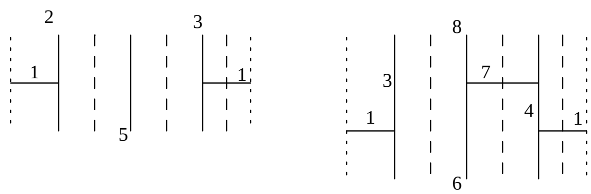

FIGURE 12. ∂V in $\{-1 < y < 1\}$

Similarly, Figures 8 and 10 show the parts of ∂V corresponding to the sub-arcs of K_1 in Figures 7 and 9. We see 4, respectively 6, points of $\Lambda \cap \mathcal{Z}_b^1$. In Figure 12, we show two of the four parts of ∂V corresponding to Figure 11, and obtain by symmetry 6 points of $\Lambda \cap \mathcal{Z}_b^1$.

The broken lines in the figures close up to three meridians (forming Λ), since the writhing number of K_1 vanishes. We have $\#(\Lambda \cap \mathcal{Z}_b^1) = 22k + 4 + 6 + 6 = 22k + 16$. The second claim of the lemma follows, since any 2–cell of $\mathcal{Z}_b$ in ∂V meets Λ in at most two arcs. □

Construction 2.9. We subdivide each 3–cell of $\mathcal{Z}_b$ that meets ∂V by inserting a copy of all 2–cells of $\mathcal{Z}_b$ in ∂V, where the copies are pairwise disjoint;[2] see Figure 13. By Lemma 2.7, we obtain a simple cellular decomposition $\mathcal{Z}_b'$ of V with

$$< 4 \cdot (48k + 56) = 192k + 224$$

vertices dual to a triangulation such that the intersection of any compact 3–cell with ∂V is connected. A representative of K_2 is formed by a path of at most $4 \cdot 11$ edges of $\mathcal{Z}_b'$.

[2]This corresponds to stellar subdivisions along some edges of the dual triangulation of $\mathcal{Z}_b$.

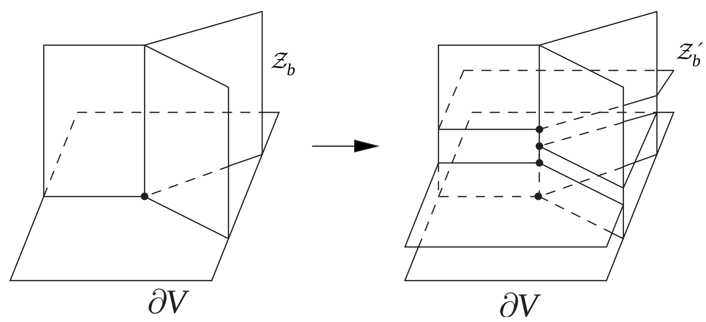

FIGURE 13. Subdividing 3–cells of $\mathcal{Z}_b$

Decompose $S^3 = V_0 \cup_{\partial V_0} (S^1 \times S^1 \times I) \cup_{\partial V_1} V_1$, where V_0, V_1 are solid tori and $S^1 \times \{*\} \times \{0\}$ (resp. $\{*\} \times S^1 \times \{1\}$) is a meridian for V_0 (resp. V_1). For $i = 0, 1$, let $\mathcal{Z}_{b,i}$ be a cellular decomposition of V_i isomorphic to $\mathcal{Z}_b'$.

Let $\Gamma \subset S^1 \times S^1$ be a union of three copies of $S^1 \times \{*\}$ and $\{*\} \times S^1$, meeting in nine points. We attach $\Gamma \times \{0\}$ and $\Gamma \times \{1\}$ at ∂V_0 and ∂V_1 as follows. Choose the three meridians in $\Gamma \times \{0\} \subset \partial V_0$ for V_0 and the three meridians in $\Gamma \times \{1\} \subset \partial V_1$ for V_1 according to Lemma 2.8. We choose the three longitudes in $\Gamma \times \{0\} \subset \partial V_0$ for V_0 (resp. in $\Gamma \times \{1\} \subset \partial V_1$ for V_1) so that each of them intersects $\mathcal{Z}_{b,0}^1$ (resp. $\mathcal{Z}_{b,1}^1$) in three points, and $\Gamma \times \{0\}$ (resp. $\Gamma \times \{1\}$) meets each 2–cell of $\mathcal{Z}_{b,0}^1$ (resp. of $\mathcal{Z}_{b,1}^1$) in at most two arcs.

We define $C = \mathcal{Z}_{b,0}^2 \cup (\Gamma \times I) \cup \mathcal{Z}_{b,1}^2 \subset S^3$. By Lemma 2.8 and by construction of Γ, Z has $< 2 \cdot (192k + 224 + 22k + 16 + 18) = 428k + 516$ intrinsic vertices. It has nine 4–valent edges, corresponding to the vertices of Γ. We perturb C along these edges, which increases the number of vertices by 18, and obtain a simple 2–polyhedron C'. □

Lemma 2.10. *The 2–polyhedron C' is the 2–skeleton of a simple cellular decomposition $\mathcal{C}_b$ of S^3 that is dual to a triangulation with $< 428k + 534$ tetrahedra.*

Proof. By Lemmas 2.7 and 2.8, the closure of each 2–stratum of Z' is a disc and the closure of any connected component of $S^3 \setminus C'$ is a ball. Thus Z' is the 2–skeleton of a simple cellular decomposition of S^3.

Let X_1, X_2 be the closures of two connected components of $S^3 \setminus C'$. There remains to show that $X_1 \cap X_2$ is connected. If $X_1 \subset V_1$ and $X_2 \subset V_2$, then $X_1 \cap X_2 = \emptyset$. If $X_1 \cup X_2 \subset S^1 \times S^1 \times I$, then $X_1 \cap X_2$ is connected by construction of Γ. According to Lemma 2.7, if $X_1 \cup X_2 \subset V_i$ for $i = 1, 2$, then $X_1 \cap X_2$ is connected. Finally, let $X_1 \subset V_i$ and $X_2 \subset S^1 \times S^1 \times I$. Then $X_1 \cap X_2$ is connected by Lemma 2.8 and by the fact that we have subdivided the 3–cells of $\mathcal{Z}_b$ that meet ∂V. □

Proof of Theorem 1.3. Let $m \in \mathbb{N}$. We set $b = (\sigma_1 \sigma_2^{-1})^m \in \mathcal{B}_4$ in order to apply the results of [5]. Let $\mathcal{C}_m = \mathcal{C}_b$ be as in Lemma 2.10; it is dual to a triangulation $\mathcal{T}_m$ of S^3 with $\leq 428k + 534 = 856m + 534$ tetrahedra.

There remains to show that $p(\mathcal{T}_m) > 2^{m-1}$. For $i = 0, 1$, let $K_{2,i} \subset \mathcal{C}_m^1$ be the copy of K_2 in V_i. Let $L_m = K_{2,0} \cup K_{2,1}$, which by Construction 2.9 is a link formed by at most 88 edges of $\mathcal{C}_m$. Let $H\colon\ S^2 \times I \to S^3$ be a $\mathcal{C}_m^1$–Morse embedding with $\mathcal{C}_m^2 \subset H(S^2 \times I)$. There is a parameter $\xi \in I$ for which H_ξ contains a meridional disc

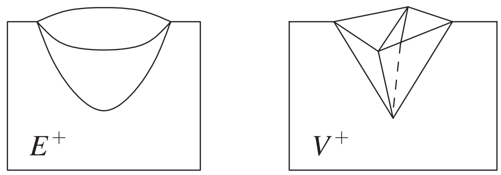

FIGURE 14. Types of critical points in $\mathcal{C}^1$

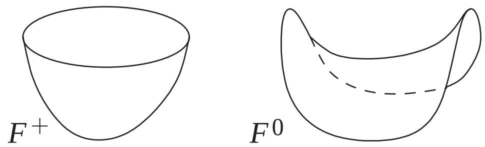

FIGURE 15. Types of critical points in a 2–cell

for V_0 or for V_1. By [5], any meridional disc in V_i intersects $K_{2,i}$ in $\geq 2^{m-1}$ points. Since $L_m \subset \mathcal{C}^1_m$ and since $\#(H_\xi \cap L_m)$ changes by ± 2 at any critical parameter, we have $p(\mathcal{T}_m) > 2^{m-1}$, which proves Theorem 1.3.

3. Contractions and expansions

This section is devoted to the proofs of Theorems 1.6 and 1.7. The proofs are outlined in the Introduction.

3.1. **Canceling pairs of critical parameters.** The setting of this subsection is more general than actually needed to prove Theorems 1.6 and 1.7. Let M be a closed orientable 3–manifold with a cellular decomposition $\mathcal{C}$ that is dual to a triangulation of M. Let S be a closed surface. The aim of this section is to deform a $\mathcal{C}^1$–Morse embedding $H\colon S \times I \to M$ by isotopy so that $c(H, \mathcal{C}^1)$ is unchanged and $c(H, \mathcal{C}^2)$ is bounded in terms of $c(H, \mathcal{C}^1)$.

A $\mathcal{C}^1$–Morse is essentially the same as an embedding of the product $S \times I$ in general position with respect to $\mathcal{C}^1$. We extend this to the notion of $\mathcal{C}^2$–Morse embeddings. The critical points in an open 2–cell are of the usual Morse types (local minimum, local maximum, or saddle point). Since general position with respect to $\mathcal{C}^2$ is not sufficient for our purposes, we further restrict the occuring types of critical points in $\mathcal{C}^1$ as in the following two definitions. Compare Figures 14 and 15.

Definition 3.1. Let $H\colon S \times I \to M$ be a $\mathcal{C}^1$–Morse embedding. Let p_0 be a critical point of H with respect to $\mathcal{C}^2$, associated to a critical parameter $\xi_0 \in I$. If there are local coordinates (x, y, z) around p_0 such that $H_\xi \cap U(p_0) = \{z = \xi - \xi_0\} \cap U(p_0)$ for $\xi \in I$, and $\mathcal{C}^2 \cap U(p_0)$ equals

- $\{z = x^2 + y^2\} \cap U(p_0)$, then p_0 (resp. ξ_0) is of type F^+.
- $\{-z = x^2 + y^2\} \cap U(p_0)$, then p_0 (resp. ξ_0) is of type F^-.
- $\{z = x^2 - y^2\} \cap U(p_0)$, then p_0 (resp. ξ_0) is of type F^0.
- $\big(\{z = x^2 + |y|\} \cup \{z \leq x^2, y = 0\}\big) \cap U(p_0)$, then p_0 (resp. ξ_0) is of type E^+.
- $\big(\{-z = x^2 + |y|\} \cup \{-z \leq x^2, y = 0\}\big) \cap U(p_0)$, then p_0 (resp. ξ_0) is of type E^-.

- $(\{z = |x| + |y|\} \cup \{z \le |x|, y = 0\} \cup \{z \ge |y|, x = 0\}) \cap U(p_0)$, then p_0 (resp. ξ_0) is of type V^+.
- $(\{-z = |x| + |y|\} \cup \{-z \le |x|, y = 0\} \cup \{-z \ge |y|, x = 0\}) \cap U(p_0)$, then p_0 (resp. ξ_0) is of type V^-.

Definition 3.2. Let $H\colon S \times I \to M$ be a $\mathcal{C}^1$–Morse embedding. If any critical point of H with respect to $\mathcal{C}^2$ is of type $F^\pm$, F^0,$E^\pm$ or $V^\pm$, then H is a $\mathcal{C}^2$–**Morse embedding**.

Note that for each critical point $p_0 \in \mathcal{C}^2$ of H corresponding to a critical parameter ξ_0, there is a *unique* open 2–cell c of $\mathcal{C}$ such that $U(p_0) \cap H_{\xi_0} \cap c \neq \emptyset$. We will use this fact in Construction 3.4 and Lemma 3.5.

Let $H\colon S \times I \to M$ be a $\mathcal{C}^2$–Morse embedding such that H_ξ splits M into two pieces for all $\xi \in I$. Let B^+, B^- be the two components of $M \setminus H(S \times I)$, with $\partial B^+ = H_1$ and $\partial B^- = H_0$. For $\xi \in I$, let $B^+(\xi)$ (resp. $B^-(\xi)$) be the closure of the component of $M \setminus H_\xi$ that contains B^+ (resp. B^-).

Our **general hypothesis** in this section is that both 0 and 1 are non-critical parameters of H with respect to $\mathcal{C}^2$, that $H_0 \cap \mathcal{C}^2 \neq \emptyset \neq H_1 \cap \mathcal{C}^2$, and that, for any open 3–cell X of $\mathcal{C}$, any circle in $H_0 \cap \partial X$ (resp. in $H_1 \cap \partial X$) bounding a disc in $B^+(0) \cap \partial X$ (resp. in $B^-(1) \cap \partial X$) also bounds a disc in $H_0 \cap X$ (resp. in $H_1 \cap X$).

We will change H by isotopy into a $\mathcal{C}^2$–Morse embedding $\tilde{H}\colon S \times I \to M$ so that $c(\tilde{H}, \mathcal{C}^2)$ is subject to an upper bound in terms of $c(H, \mathcal{C}^1)$. Whenever H has a critical point of type $F^\pm$, then it "cancels" with a critical point of type F^0 (see Lemma 3.3). We change H according to Figures 16 and 17, removing the canceling pair of critical points (see Lemma 3.5).

Let $p_0 \in \mathcal{C}^0$ be a critical point of type F^+ that belongs to a critical parameter ξ_0 of H. There is a unique open 3–cell X of $\mathcal{C}$ such that $U(p_0) \cap H_{\xi_0} \cap X \neq \emptyset$.

Lemma 3.3. (1) *There is a non-empty open interval $(\xi_0, \overline{\xi_0})$ such that for any $\xi \in (\xi_0, \overline{\xi_0})$ there is a connected component $\gamma(\xi)$ of $H_\xi \cap \partial X$ that is a circle, such that $\gamma(\xi)$ varies continously in ξ and $\lim_{\xi \to \xi_0} \gamma(\xi) = p_0$.*

(2) *If $\overline{\xi_0}$ is maximal, then $\gamma(\overline{\xi_0}) = \lim_{\xi \nearrow \overline{\xi_0}} \gamma(\xi)$ is a circle that is not a connected component of $H_{\overline{\xi_0}} \cap \partial X$ and contains a critical point $\overline{p_0}$ of H of type F^0.*

Proof. For the first part of the lemma, let $\epsilon > 0$ be small enough such that there is no critical parameter of H with respect to $\mathcal{C}^2$ in $(\xi_0, \xi_0 + \epsilon)$. Then, $H_{\xi_0 + \epsilon} \cap \partial X$ contains a small circle $\gamma(\xi_0 + \epsilon)$ around p_0. For $\xi \in (\xi_0, \xi_0 + \epsilon)$, one obtains $\gamma(\xi)$ from $\gamma(\xi_0 + \epsilon)$ by isotopy induced by H.

For the second part of the lemma, note that $\gamma(\xi_0 + \epsilon)$ bounds a disc in $\partial X \cap B^-(H_{\xi_0 + \epsilon})$. By induction, $\gamma(\overline{\xi_0})$ bounds a disc in $\partial X \cap B^-(H_{\overline{\xi_0}})$ and does not bound a disc in $H_{\overline{\xi_0}} \cap X$. Thus by the general hypothesis, $\overline{\xi_0} < 1$. By maximality of $\overline{\xi_0}$, either $\gamma(\overline{\xi_0})$ is a critical point of H of type V^-, E^- or F^-, or $\gamma(\overline{\xi_0})$ is a circle containing a critical point of H of type F^0. Since $H_0 \cap \mathcal{C}^2 \neq \emptyset$ by our general hypothesis, it follows by induction that the connected component of $H_\xi \cap X$ containing $\gamma(\xi)$ in its boundary is not a disc, for $\xi \in (\xi_0, \overline{\xi_0})$. Thus the critical point in $\gamma(\overline{\xi_0})$ is of type F^0. □

We keep the notations of the preceding lemma and assume that $\overline{\xi_0}$ is maximal. Let $\xi_0 < \xi_1 < \cdots < \xi_k = \overline{\xi_0}$ be the critical parameters of H with respect to $\mathcal{C}^2$ in the closed interval $[\xi_0, \overline{\xi_0}]$, corresponding to the critical points $p_0, p_1, \ldots, p_k$.

Construction 3.4 below yields two compact arcs $\alpha_1, \alpha_2 \subset \partial X$ so that for $\xi \in [\xi_0', \xi_k]$ and $m = 1, 2$ holds

(1) $\alpha_m \cap H_\xi$ is empty or a single point $\alpha_m(\xi)$,
(2) $\alpha_1 \cup \alpha_2$ contains at most one critical point of H of type F^+,
(3) $\alpha_m \cap \mathcal{C}^1$ consists of critical points of H of type E^+ and V^+, and
(4) $\alpha_2(\xi) \in \gamma(\xi)$, and either $\alpha_1(\xi) \notin \gamma(\xi)$ or $\alpha_1(\xi) = \alpha_2(\xi) = p_k$.

Construction 3.4. We construct α_1, α_2 iteratively. Let

$$\alpha_1 \cap B^+(\overline{\xi_0}) = \alpha_2 \cap B^+(\overline{\xi_0}) = p_k.$$

Let $i \in \{0, 1, \ldots, k\}$. For both $m = 1, 2$, suppose that $\alpha_m \cap H(S^2 \times [\xi_i, \xi_k])$ is already constructed. If $\alpha_m(\xi_i)$ is a critical point of H of type F^+, then we stop the construction and define $\xi_0' = \xi_i$. Otherwise, by construction either $\alpha_m(\xi_i) \notin \mathcal{C}^1$ or $\alpha_m(\xi_i)$ is a critical point of H of type E^+ or V^+. Thus by definition of $\mathcal{C}^2$–Morse embeddings, there is a unique open 2–cell $c_m \subset \partial X$ of $\mathcal{C}$ with $U(\alpha_m(\xi_i)) \cap H_{\xi_i} \cap c_m \neq \emptyset$. We extend α_m by an arc in $\overline{c_m} \cap H\,(S \times [\xi_{i-1}, \xi_i])$, so that

(1) $\alpha_1(\xi) \notin \gamma(\xi)$ and $\alpha_2(\xi) \in \gamma(\xi)$ for $\xi \in (\xi_{i-1}, \xi_i)$,
(2) if $\alpha_m(\xi_{i-1}) \in \mathcal{C}^1$, then $\alpha_m(\xi_{i-1}) = p_{i-1}$, and
(3) $\alpha_m(\xi_{i-1}) = p_{i-1}$ if and only if $U(\alpha_m(\xi_{i-1})) \cap H_{\xi_{i-1}} \cap c_m = \emptyset$.

This is possible, since if $\alpha_2(\xi_i) \in \gamma(\xi_i)$, then we can stay in $\gamma(\xi)$, if $\alpha_m(\xi_{i-1})$ is not a critical point, then we can avoid to run into $\mathcal{C}^1$, and if $U(p_{i-1}) \cap H_{\xi_{i-1}} \cap c_m \neq \emptyset$, then we can avoid to run into the critical point p_{i-1}. It follows by property (c) that $\alpha(\xi_{i-1}) = p_{i-1}$ only if p_{i-1} is of type F^+, E^+ or V^+. □

Lemma 3.5. *There is a $\mathcal{C}^2$-Morse embedding $\widetilde{H}\colon S \times I \to M$ isotopic to H without critical points of type $F^\pm$ such that $\widetilde{H}(S \times \{0,1\}) = H(S \times \{0,1\})$ and $c(\widetilde{H}, \mathcal{C}^1) = c(H, \mathcal{C}^1)$.*

Proof. Let $\xi_0 \in I$ be a critical parameter of H of type F^+, and let p_0 be the corresponding critical point. We use the notations of Construction 3.4. If $\xi_0 < \xi_0'$, then we replace p_0 by the critical point of H of type F^+ that corresponds to the critical parameter ξ_0'. Since there are only finitely many critical parameters in $[\xi_0, \xi_0']$, we can choose p_0 so that $\xi_0 = \xi_0'$, by repeating this process. Thus we can assume that $\alpha_2(\xi_0) = p_0$.

Let (x, y, z) be local coordinates around p_k as in Definition 3.1. Let $\epsilon > 0$ and $U(p_k)$ such that $H_{\xi_k - \epsilon} \cap U(p_k)$ is a disc, and ξ_0 (resp. ξ_k) is the only critical parameter of H with respect to $\mathcal{C}^2$ in $[\xi_0 - \epsilon, \xi_0 + \epsilon]$ (resp. $[\xi_k - \epsilon, \xi_k + \epsilon]$). Define $D' = \{-\epsilon \le z \le -y^2, x = 0\} \subset U(p_k)$. We have $\partial D' \subset H_{\xi_k - \epsilon} \cup \mathcal{C}^2$. There is a compact disc $D \subset X \cap B^-(H_{\xi_k - \epsilon})$ such that $D \cap H_\xi$ is a copy of $\gamma(\xi)$, for all $\xi \in (\xi_0, \xi_k - \epsilon]$.

Let $\alpha_1, \alpha_2 \subset \partial X$ be as in Construction 3.4. We change H into an embedding $\widetilde{H}\colon S \times I \to M$ in the following way; compare Figures 16 and 17:

(1) For $\xi \in [0, \xi_0 - \epsilon]$, let $\widetilde{H}(\cdot, \xi) \equiv H(\cdot, \xi)$.
(2) When the parameter ξ increases from $\xi_0 - \epsilon$ to ξ_0, $\widetilde{H}$ pushes a finger along $\alpha_1 \cup \alpha_2$ towards p_0, so that $\widetilde{H}_{\xi_0} = \partial\left((B^-(\xi_0) \setminus U(D)) \cup U(\alpha_1 \cup \alpha_2)\right)$.
(3) For $\xi \in [\xi_0, \xi_k - \epsilon]$, set $\widetilde{H}_\xi = \partial\left((B^-(\xi) \setminus U(D)) \cup U(\alpha_1 \cup \alpha_2)\right)$.
(4) In $[\xi_k - \epsilon, \xi_k + \epsilon]$, $\widetilde{H}$ induces an isotopy mod $\mathcal{C}^2$ with support in $U(D' \cup D)$, relating $\widetilde{H}_{\xi_k - \epsilon}$ with $\widetilde{H}_{\xi_k + \epsilon} = H_{\xi_k + \epsilon}$.
(5) For $\xi \in [\xi_k + \epsilon, 1]$, let $\widetilde{H}(\cdot, \xi) \equiv H(\cdot, \xi)$.

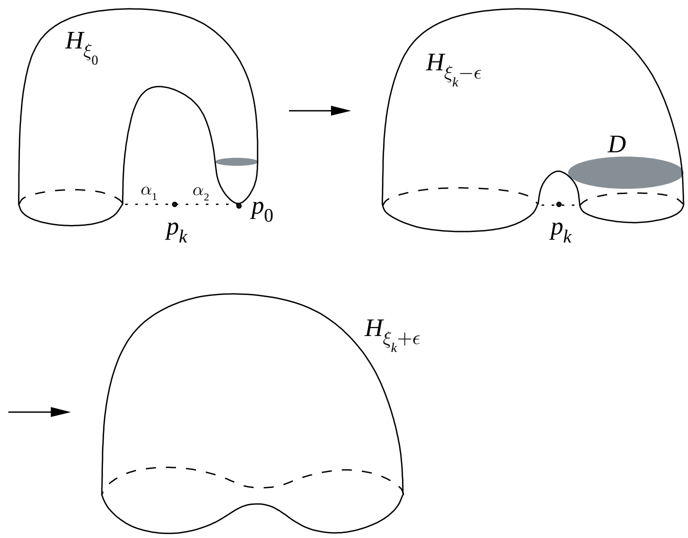

FIGURE 16. The moves of H

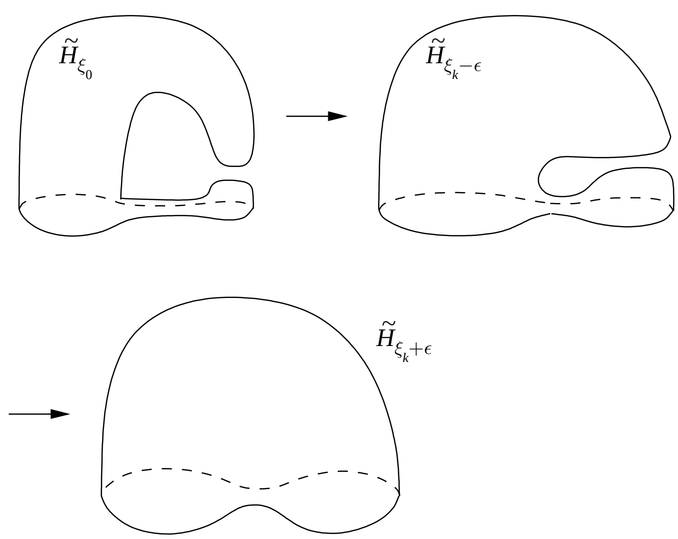

FIGURE 17. The moves of $\widetilde{H}$

When we push a finger along $\alpha_1 \cup \alpha_2$, critical points of $\widetilde{H}$ of type E^+ or V^+ occur at $(\alpha_1 \cup \alpha_2) \cap \mathcal{C}^1$. By Construction 3.4, these are also critical points of H of type E^+ and V^+. Critical points of $\widetilde{H}$ in $\mathcal{C}^1 \cap \partial U(\alpha_1 \cup \alpha_2)$ do not occur.[3] It follows

[3]This would be the case if $(\alpha_1 \cup \alpha_2) \cap \mathcal{C}^1$ would contain non-critical points or critical points of type E^- or V^-.

that the critical points of H and $\widetilde{H}$ in $\mathcal{C}^1$ coincide, $c(\widetilde{H}, \mathcal{C}^1) = c(H, \mathcal{C}^1)$, although the order of the corresponding critical *parameters* has changed.

One sees that $\widetilde{H}$ has exactly two critical points with respect to $\mathcal{C}^2$ less than H, namely p_0 and p_k. One iterates this construction and removes all critical points of type F^+ of H. By the symmetric construction, one also removes all critical points of type F^- of H, and the lemma follows. □

After getting rid of the critical points of type $F^\pm$, there remains to estimate the number of critical points of type F^0.

Lemma 3.6. *Assume that H has no critical points of type $F^\pm$. Then H has $\leq \chi(B^- \cap \mathcal{C}^2) + \chi(B^+ \cap \mathcal{C}^2) - \chi(\mathcal{C}^2) + \chi(\mathcal{C}^0) + c(H, \mathcal{C}^1)$ critical points of type F^0.*

Proof. For any $\xi \in I$, define $P_\xi = \left(\mathcal{C}^2 \cap B^+(\xi)\right) \cup H_\xi$. The homeomorphism type of P_ξ changes only at critical parameters of H with respect to $\mathcal{C}^2$. Let ξ_0 be a critical parameter of H with respect to $\mathcal{C}^2$. We choose $\epsilon > 0$ so that ξ_0 is the only critical parameter of H in the interval $[\xi_0 - \epsilon, \xi_0 + \epsilon]$. Denote by χ^+, χ^- the Euler characteristic of $P_{\xi_0+\epsilon}, P_{\xi_0-\epsilon}$. We have

(1) $\chi^+ = \chi^-$, if ξ_0 is of type E^+ or V^+,
(2) $\chi^+ = \chi^- - 1$, if ξ_0 is of type E^-
(3) $\chi^+ = \chi^- - 2$, if ξ_0 is of type V^-, and
(4) $\chi^+ = \chi^- + 1$, if ξ_0 is of type F^0.

Since $\chi(P_0) = \chi(\mathcal{C}^2) + \chi(S) - \chi(B^- \cap \mathcal{C}^2)$ and $\chi(P_1) = \chi(S) + \chi(B^+ \cap \mathcal{C}^2)$, and since H has at most $\#(\mathcal{C}^0) = \chi(\mathcal{C}^0)$ critical points of type V^-, the lemma follows. □

3.2. **Relating triangulations by contractions and expansions.** This subsection is devoted to the proof of Theorem 1.7. We start with the construction of a triangulation associated to embedded fake surfaces, so that the triangulation changes by contractions (resp. expansions) if the fake surface changes by deletion (resp. insertion) of 2–cells. The proof of Theorem 1.7 is an application of this construction to fake surfaces that are associated to the non-critical parameters of a $\mathcal{C}^2$–Morse embedding.

A fake surface $Q \subset S^3$ is **regular** if it is the 2–skeleton of a simple cellular decomposition $\mathcal{C}$ of S^3 such that any 2–cell of $\mathcal{C}$ is contained in the boundary of two different 3–cells of $\mathcal{C}$. It follows easily that the closure of any open 3–cell of $\mathcal{C}$ is a compact ball, and the barycentric subdivision of $\mathcal{C}$ is a triangulation of S^3 (i.e., it has no multiple edges). Since $\mathcal{C}$ and its barycentric subdivision are determined by $Q \subset S^3$, we denote this triangulation by $\mathcal{T}(Q)$. The next lemma provides conditions under which the deletion of a 2–stratum of Q gives rise to a sequence of contractions of $\mathcal{T}(Q)$.

Lemma 3.7. *Let $Q_1, Q_2 \subset S^3$ be regular fake surfaces. Let c be a 2–stratum of Q_1 whose closure contains k intrinsic vertices. If Q_2 is obtained from Q_1 by deletion of c, then $\mathcal{T}(Q_2)$ is obtained from $\mathcal{T}(Q_1)$ by a sequence of $4k + 2$ contractions.*

Proof. By hypothesis on Q_1, it is the 2–skeleton of a simple cellular decomposition $\mathcal{C}_1$ of S^3, and c is a 2–cell contained in the boundary of two different 3–cells X_1, X_2 of $\mathcal{C}_1$. We contract $\mathcal{T}(Q_1)$ along the edges that connect the barycenter of c with the barycenters of X_1, X_2. By hypothesis on Q_2, the closure of any connected component of $S^3 \setminus Q$ is a ball, hence $\partial X_1 \cap \partial X_2 = \overline{c}$. Thus the two contractions are allowed, i.e., do not introduce multiple edges.

Any edge $e \subset \partial c$ is adjacent to exactly two 2–cells c_1, c_2 of $\mathcal{C}_1$ that are different from c. By hypothesis on Q_2, we have $\partial c_1 \cap \partial c_2 = \overline{e}$. Thus we can further contract along the edges of $\mathcal{T}(Q_1)$ that connect the barycenter of e with the barycenters of c_1, c_2, without introducing multiple edges.

Any vertex $v \in \partial c$ is endpoint of exactly two edges e_1, e_2 of $\mathcal{C}_1$ that are not contained in ∂c. Since $\partial e_1 \cap \partial e_2 = v$, we can further contract along the edges of $\mathcal{T}(Q_1)$ that connect v with the barycenters of e_1, e_2. These $2+2k+2k$ contractions yield $\mathcal{T}(Q_2)$. □

Our plan is to associate fake surfaces to the non-critical parameters of a $\mathcal{C}^2$–Morse embedding, and apply to them the preceding techniques. For this aim, we need a $\mathcal{C}^2$–Morse embedding in a particularly nice position, provided by the following lemma.

Lemma 3.8. *Let $\mathcal{T}$ be a triangulation of S^3, and let $\mathcal{C}$ be its dual cellular decomposition. There are two vertices $v_0, v_1 \in \mathcal{C}^0$ and a $\mathcal{C}^2$–Morse embedding $H\colon S^2 \times I \to S^3$ with $c(H, \mathcal{C}^1) \leq p(\mathcal{T}) + 2n + 2$ and $H(S^2 \times I) = S^3 \setminus U(\{v_0, v_1\})$,*

Proof. By definition, there is a $\mathcal{C}^1$–Morse embedding $H'\colon S^2 \times I \to S^3$ with $\mathcal{C}^2 \subset H'(S^2 \times I)$ and $c(H', \mathcal{C}^1) = p(\mathcal{T})$. Let X_0, X_1 be the open 3–cells of $\mathcal{C}$ that contain H'_0, H'_1. Pick two different vertices v_0, v_1 in $\partial X_0, \partial X_1$. We change H' into a $\mathcal{C}^1$–Morse embedding $H''\colon S^2 \times I \to S^3$ by pushing a finger from H'_0 towards v_0 and from H'_1 towards v_1, so that $\mathcal{C}^2 \subset H''(S^2 \times I)$ and v_0 (resp. v_1) are the critical points of H'' that correspond to the smallest (resp. biggest) critical parameter of H'' with respect to $\mathcal{C}^2$, denoted by ξ_0 (resp. ξ_1). This is possible by introducing at most one critical point in each of the eight edge germs at v_0, v_1. Thus $c(H'', \mathcal{C}^1) \leq p(\mathcal{T}) + 8$.

By a small isotopy, we factorize the critical points of H'' in $\mathcal{C}^2$, except v_0, v_1, by critical points of type $F^\pm$, F^0, $E^\pm$ and $V^\pm$. Any critical point of H'' in the interior of an edge of $\mathcal{C}$ factorizes by one critical point of type $E^\pm$ and some other critical points in $\mathcal{C}^2 \setminus \mathcal{C}^1$. Let $v_2 \in \mathcal{C}^0 \setminus \{v_0, v_1\}$ be a vertex, corresponding to a critical parameter ξ_2 of H''. If a component of $U(v_2) \setminus U(H''_{\xi_2})$ intersects $\mathcal{C}^1$ in exactly one (resp. two) arcs, then the critical point v_2 factorizes by one (resp. two) critical points of type $E^\pm$, one critical point of type $V^\mp$ (with consistent signs), and some critical points outside $\mathcal{C}^1$. Thus we obtain a $\mathcal{C}^2$–Morse embedding $H'''\colon S^2 \times I \to S^3$ with $c(H''', \mathcal{C}^1) \leq p(\mathcal{T}) + 2n + 4$. Now, a $\mathcal{C}^2$–Morse embedding $H\colon S^2 \times I \to S^3$ with the claimed properties is given by the restriction of H''' to $S^2 \times [\xi_0 + \epsilon, \xi_1 - \epsilon]$, for small $\epsilon > 0$. □

To prove Theorem 1.7, it suffices to show that any triangulation $\mathcal{T}$ of S^3 can be transformed into the barycentric subdivision of the boundary complex of a 4–simplex by $\leq 325p(\mathcal{T}) + 254$ expansions and contractions. Let $\mathcal{C}$, v_0, v_1 and $H\colon S^2 \times I \to S^3$ be as in the preceding lemma, with $H_0 = \partial U(v_0)$ and $H_1 = \partial U(v_1)$. For $\xi \in I$, let $B^+(\xi)$ (resp. $B^-(\xi)$) be the closure of the component of $S^3 \setminus H_\xi$ that contains v_1 (resp. v_0).

Since $H_0 \setminus \mathcal{C}^2$ and $H_1 \setminus \mathcal{C}^2$ are disjoint unions of discs, H satisfies the general hypothesis of Subsection 3.1. Hence by Lemma 3.5, we can assume that H has no critical points of type $F^\pm$. By Lemma 3.6, H has at most $c(H, \mathcal{C}^1) + 2n + 2$ critical points of type F^0 (hint: we have $\chi(B^-(0) \cap \mathcal{C}^2) = \chi(U(v_0) \cap \mathcal{C}^2) = 1$, $\chi(B^+(1) \cap \mathcal{C}^2) = 1$, and $-\chi(\mathcal{C}^2) = \chi(S^3 \setminus \mathcal{C}^2) \leq n$). With Lemma 3.8, we have $c(H, \mathcal{C}^2) \leq 2p(\mathcal{T}) + 6n + 6 \leq 8p(\mathcal{T}) + 6$.

We define $P_\xi = \big(\mathcal{C}^2 \cap B^+(\xi)\big) \cup H_\xi$ for $\xi \in I$. By the next lemma, the triangulations $\mathcal{T}_\xi = \mathcal{T}(P_\xi)$ of S^3 are defined for any non-critical parameter $\xi \in I$ of H with respect to $\mathcal{C}^2$.

Lemma 3.9. *For any non-critical parameter $\xi \in I$, P_ξ is a regular fake surface.*

Proof. Let c be a 2–cell of $\mathcal{C}$ and assume that some component γ of $c \cap H_\xi$ is a circle. It bounds a disc $D \subset c$. Let a collar of γ in D be contained in $B^+(\xi)$ (resp. in $B^-(\xi)$). Since H has no critical parameters of type F^- (resp. F^+), it follows by induction on the number of critical parameters of H that $c \cap H_1$ (resp. $c \cap H_0$) contains a circle, in contradiction to the hypothesis on H. Thus H_ξ intersects the 2–cells of $\mathcal{C}$ in arcs.

Similarly one shows that $H_\xi \setminus \mathcal{C}^2$ is a disjoint union of open discs, and $S^3 \setminus P_\xi$ is a disjoint union of open 3–balls. Thus the open 2–strata of P_ξ are discs. Since both H_0 and H_1 intersect $\mathcal{C}^1$ and $\mathcal{C}^1$ is connected, P_ξ has an intrinsic vertex in $H_\xi \cap \mathcal{C}^1$. Since ξ is not a critical parameter of H, P_ξ is simple. In conclusion, P_ξ is the 2–skeleton of a simple cellular decomposition $\mathcal{C}_\xi$ of S^3.

Let c be a 2–cell of $\mathcal{C}_\xi$. If $c \subset \mathcal{C}^2$, then it separates two 3–cells of $\mathcal{C}_\xi$, since $\mathcal{C}$ is dual to a triangulation. If $c \subset H_\xi$, then it separates the 3–cell of $\mathcal{C}_\xi$ corresponding to $B^-(\xi)$ from another 3–cell of $\mathcal{C}_\xi$. □

We show how P_ξ and $\mathcal{T}_\xi$ change when ξ passes a critical parameter ξ_0 of H with respect to $\mathcal{C}^2$. Let $p_0 \in \mathcal{C}^2$ be the critical point corresponding to ξ_0. We choose $\epsilon > 0$ so that ξ_0 is the only critical parameter in $[\xi_0 - \epsilon, \xi_0 + \epsilon]$. Choose local coordinates (x, y, z) around p_0 as in Definition 3.1. Let $r > 0$ be small enough such that $B = \{x^2 + y^2 + z^2 \leq r^2\}$ is a closed regular neighborhood of p_0.

By isotopy of $H_{\xi_0 \pm \epsilon}$ mod $\mathcal{C}^2$, we can assume that $B \cap H_{\xi_0 - \epsilon} = D$ and $B \cap H_{\xi_0 + \epsilon} = D'$ are discs, $\partial B = D \cup D'$, and $H_{\xi_0+\epsilon} = (H_{\xi_0-\epsilon} \setminus D) \cup D'$. We define $P' = (P_{\xi_0-\epsilon} \setminus B) \cup \partial B$. One easily verifies that P' is a regular fake surface, hence $\mathcal{T}(P')$ is defined. By deletion of its 2–stratum D, one obtains $P_{\xi_0+\epsilon}$. In ∂D are at most 4 intrinsic vertices (namely when p_0 is of type F^0). Thus by Lemma 3.7, $\mathcal{T}_{\xi_0+\epsilon}$ is obtained from $\mathcal{T}(P')$ by ≤ 18 contractions. We consider how P' and$P_{\xi_0-\epsilon}$ are related by deletions of 2–strata.

(1) If p_0 is of type F^0, then one obtains $P_{\xi_0-\epsilon}$ from P' up to isotopy by deletion of the two 2–strata corresponding to $D' \cap \{z \leq x^2 - y^2\}$. They both have 2 intrinsic vertices in its boundary.
(2) If p_0 is of type E^+, then one obtains $P_{\xi_0-\epsilon}$ from P' up to isotopy by deletion of the 2–stratum corresponding to $D' \cap \{z \leq x^2 + y, y \geq 0\}$. It has 4 intrinsic vertices in its boundary.
(3) If p_0 is of type E^-, then $P_{\xi_0-\epsilon}$ is isotopic to P'.
(4) If p_0 is of type V^+, then one obtains $P_{\xi_0-\epsilon}$ from P' up to isotopy by deletion of the 2–stratum corresponding to $D' \cap \{z \leq |x| + y, y \geq 0\}$. It has 5 intrinsic vertices in its boundary.
(5) If p_0 is of type V^-, then one obtains $P_{\xi_0-\epsilon}$ from P' up to isotopy by insertion of the 2–stratum corresponding to $B \cap \{z \leq -|y|, x = 0\}$. It has 3 intrinsic vertices in its boundary.

Thus by Lemma 3.7, $\mathcal{T}(P')$ is obtained from $\mathcal{T}_{\xi_0-\epsilon}$ by ≤ 22 successive contractions or expansions.

With our bound for $c(H, \mathcal{C}^2)$, it follows that $\mathcal{T}_0$ and $\mathcal{T}_1$ are related by

$$\leq (18 + 22) \cdot (8p(\mathcal{T}) + 6) = 320p(\mathcal{T}) + 240$$

expansions and contractions. Let n be the number of tetrahedra of $\mathcal{T}$. Since $\mathcal{T}$ has $2n$ 2–simplices and at most $2n$ edges, one obtains its barycentric subdivision $\mathcal{T}(\mathcal{C}^2)$ by $\leq 5n \leq 5p(\mathcal{T})$ expansions. Since Q_0 is isotopic to the result of adding one triangular 2–stratum to $\mathcal{C}^2$, $\mathcal{T}(\mathcal{C}^2)$ can be transformed into $\mathcal{T}_0$ by 14 expansions. We count together and find that $\mathcal{T}$ can be transformed into $\mathcal{T}_1$, the barycentric subdivision of the boundary complex of a 4–simplex, by $\leq 5p(\mathcal{T}) + 14 + 320p(\mathcal{T}) + 240 = 325p(\mathcal{T}) + 254$ contractions and expansions. This yields Theorem 1.7.

3.3. **How to make a triangulation edge contractible.** In this subsection, we prove Theorem 1.6. We use the same notations as in the previous subsection. The idea for the proof is to insert 2–strata as above in the transition from $P_{\xi_0-\epsilon}$ to P', and to postpone the deletion of 2–strata until all insertions are done. This means to transform $\mathcal{T}$ by expansions into a triangulation that is then turned into the boundary complex of a 4–simplex by a sequence of contractions.

Let $\xi_1 < \xi_2 < \cdots < \xi_Z$ be the critical parameters of H with respect to $\mathcal{C}^2$, let $p_1, \ldots, p_Z \in \mathcal{C}^2$ be the corresponding critical points, and let $\xi_{Z+1} = 1$. Let $\epsilon > 0$ be small enough such that ξ_i is the only critical parameter of H in $[\xi_i - \epsilon, \xi_i + \epsilon]$, for all $i = 1, \ldots, Z$. For any i, choose local coordinates (x_i, y_i, z_i) around p_i as in Definition 3.1. Let $r > 0$ be small enough such that $B_i = \{x_i^2 + y_i^2 + z_i^2 \leq r^2\}$ is a closed regular neighborhood of p_i. We arrange $H_{\xi_i \pm \epsilon}$ by isotopy mod $\mathcal{C}^2$ so that $\partial B_i \cap H_{\xi_i - \epsilon} = D_i$ and $\partial B_i \cap H_{\xi_i + \epsilon} = D_i'$ are discs, $\partial B_i = D_i \cup D_i'$, and $H_{\xi_i+\epsilon} = (H_{\xi_i-\epsilon} \setminus D_i) \cup D_i'$.

We now define iteratively a sequence $Q_1, \ldots, Q_{Z+1} \subset S^3$ of regular fake surfaces, together with graphs $\Gamma_i \subset Q_i^1$, so that $P_{\xi_i-\epsilon} \subset Q_i$ and $\Gamma_i = \partial(Q_i \setminus P_{\xi_i-\epsilon})$. We define $Q_1 = (\mathcal{C}^2 \cap B^+(\xi_1 - \epsilon)) \cup H_{\xi_1-\epsilon} = P_{\xi_1-\epsilon}$ and $\Gamma_1 = \emptyset$. Let $i \in \{1, \ldots, Z\}$. We arrange $\Gamma_i \subset Q_i^1$ by an isotopy of Q_i mod $\mathcal{C}^2$ so that Γ_i intersects ∂D_i transversely and $\#(\Gamma_i \cap \partial D_i)$ is as small as possible. If ξ_i is not of type V^-, then we define $Q_i' = (Q_i \setminus B_i) \cup \partial B_i$ and $\Gamma_i' = (\Gamma_i \setminus D_i) \cup \partial D_i$. If ξ_i is of type V^-, then we define $Q_i' = (Q_i \setminus B_i) \cup \partial B_i \cup (\{x_i = 0\} \cap B_i)$ and $\Gamma_i' = (\Gamma_i \setminus D_i) \cup \partial D_i \cup (\{x_i = 0\} \cap D_i')$; here Q' is isotopic to Q_i. We define Q_{i+1} and Γ_{i+1} as the result of Q_i' and Γ_i' under an isotopy mod $\mathcal{C}^2$ that relates $H_{\xi_i+\epsilon}$ with $H_{\xi_{i+1}-\epsilon}$.

It follows as in the proof of Lemma 3.9 that $Q_1, \ldots, Q_{Z+1}$ are regular fake surfaces. Let $\mathcal{T}_i = \mathcal{T}(Q_i)$, for $i = 1, \ldots, Z + 1$. In the following two lemmas, we show that for $i = 1, \ldots, Z$ one obtains Q_i from Q_{i+1} by deletion of 2–strata, with an estimate for the number of vertices in the boundary of these 2–strata.

Lemma 3.10. *For $i = 1, \ldots, Z$, we have $\#(\Gamma_i \cap \mathcal{C}^2) \leq 4(i - 1)$.*

Proof. Let $j \geq 1$. One observes that $\Gamma_j' \cap \mathcal{C}^2$ comprises at most four points more than $\Gamma_j \cap \mathcal{C}^2$, namely the points of $\partial D_i \cap \mathcal{C}^2$. The claim follows by induction, with $\Gamma_1 = \emptyset$. □

Lemma 3.11. *For $i = 1, \ldots, Z$, one obtains Q_i from Q_{i+1} up to isotopy by deletion of one 2–stratum with at most 5 vertices in its boundary, or by deletion of two 2–strata of Q_{i+1} with at most $2i$ vertices in its boundary.*

Proof. Let p_i not be of type F^0. Then any simple arc in $D_i \setminus \mathcal{C}^2$ with boundary in ∂D_i is parallel in $D_i \setminus \mathcal{C}^2$ to a sub-arc of $\partial D_i \setminus \mathcal{C}^2$. Thus $U(p_i) \cap \Gamma_i = \emptyset$ by the minimality of $\#(\Gamma_i \cap \partial D_i)$. If p_i is of type V^-, then Q_i is isotopic to Q_{i+1}. If p_i is of type $E^\pm$ or V^+, then we apply the analysis in the proof of Theorem 1.7 and see that one obtains Q_i from $Q_i' \simeq Q_{i+1}$ by deletion of one 2–stratum with at most 5 vertices in the boundary.

If p_i is of type F^0, then one obtains Q_i from $Q_i' \simeq Q_{i+1}$ by deletion of the two connected components of $D_i' \cap \{z_i \le x_i^2 - y_i^2\}$. We estimate the number of intrinsic vertices in the boundary of these 2–strata. Let c be the 2–stratum of Q_i that contains $D_i \cap \{z_i \le x_i^2 - y_i^2\}$. By minimality of $\Gamma_i \cap \partial D_i$, there are at most $\frac{1}{2}\#(\Gamma_i \cap \partial c)$ arcs in $\Gamma_i \cap D_i$, each connecting the two components of $c \cap \partial D_i$. Thus by Lemma 3.10 both components of $D_i' \cap \{z_i \le x_i^2 - y_i^2\}$ have at most $2(i-1)+2$ vertices in their boundary. $\square$

By Lemmas 3.7 and 3.11, one obtains $\mathcal{T}_{i+1}$ from $\mathcal{T}_i$ by at most $\max\{22, 16i+4\}$ expansions. As in the previous section, $\mathcal{T}_1$ is obtained from $\mathcal{T}$ by $\le 5p(\mathcal{T})+14$ expansions, and $Z \le 8p(\mathcal{T})+6$. Thus one obtains $\mathcal{T}_{Z+1}$ from $\mathcal{T}$ by

$$\begin{aligned} &\le \quad 5p(\mathcal{T}) + 14 + 22 + \sum_{i=2}^{Z}(16i+4) \\ &\le \quad 512(p(\mathcal{T}))^2 + 869p(\mathcal{T}) + 376 \end{aligned}$$

expansions.

One obtains $(\mathcal{C}^2 \cap B^+(1-\epsilon)) \cup H_{1-\epsilon} = P_{1-\epsilon}$ from Q_{Z+1} by successive deletion of the 2–strata $D_1, D_2, \dots, D_Z$. One checks that these deletions satisfy the hypothesis of Lemma 3.7. Thus $\mathcal{T}(P_{1-\epsilon})$ is the result of $\mathcal{T}_{Z+1}$ under successive contractions. Since $H_{\xi_Z+\epsilon}$ is isotopic mod $\mathcal{C}^2$ to $\partial U(v_1)$, $\mathcal{T}(P_{\xi_Z+\epsilon})$ is the barycentric subdivision of the boundary complex of a 4–simplex. Thus $\mathcal{T}_{Z+1}$ is edge contractible, which finally proves Theorem 1.6.

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Department of Mathematics, Darmstadt University of Technology, Schlossgartenstr. 7, 64289 Darmstadt, Germany

E-mail address: king@mathematik.tu-darmstadt.de